

# Torsion-freeness for fusion rings and tensor $C^*$ -categories

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## Abstract

Torsion-freeness for discrete quantum groups was introduced by R. Meyer in order to formulate a version of the Baum-Connes conjecture for discrete quantum groups. In this note, we introduce torsion-freeness for abstract fusion rings. We show that a discrete quantum group is torsion-free if its associated fusion ring is torsion-free. In the latter case, we say that the discrete quantum group is *strongly* torsion-free. As applications, we show that the discrete quantum group duals of the free unitary quantum groups are strongly torsion-free, and that torsion-freeness of discrete quantum groups is preserved under Cartesian and free products. We also discuss torsion-freeness in the more general setting of abstract rigid tensor  $C^*$ -categories.

## Introduction

R. Meyer introduced in [17] the notion of torsion-freeness for discrete quantum groups in order to state a plausible Baum-Connes conjecture for them. In his definition, a discrete quantum group is torsion-free if and only if any

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action of its dual compact quantum group on a finite-dimensional  $C^*$ -algebra is equivariantly Morita equivalent to a trivial action.

For the duals of the  $q$ -deformations of semi-simple compact Lie groups, torsion-freeness was shown in [12], based on the fundamental computation in [26] for the dual of quantum  $SU(2)$  [30]. By means of monoidal equivalence [6], this result was then extended to certain free orthogonal quantum groups introduced in [24]. However, up till now it was not known if also the free unitary quantum groups are torsion-free [25].

The main result of this note is that the duals of free unitary quantum groups are indeed torsion-free. Our result will in fact show that they are torsion-free in a strong way. For this, recall that to any discrete quantum group can be associated a *fusion ring*, which remembers how tensor products of irreducible representations of its dual compact quantum group decompose into irreducibles. We then propose a definition of torsion-freeness for fusion rings in terms of *cofinite fusion modules*, and show that any discrete quantum group with a torsion-free fusion ring is automatically torsion-free. We hence call a discrete quantum group *strongly* torsion-free if its associated fusion ring is torsion-free. By a combinatorial argument, it is shown that the duals of free unitary quantum groups are strongly torsion-free.

The notion of (strong) torsion-freeness can in fact be captured purely within the language of rigid tensor  $C^*$ -categories. Although the case of discrete quantum groups can be subsumed in this more general setting, we treat it separately as to be more accessible to people unfamiliar with tensor  $C^*$ -categories.

The structure of this paper is as follows. In the *first section*, we define torsion-freeness for fusion rings, and show that the fusion rings coming from free unitary quantum groups are torsion-free. We also show that torsion-freeness is preserved by free products, and discuss the case of tensor products. In the *second section*, we show torsion-freeness of discrete quantum groups with torsion-free fusion ring, and deduce that the duals of free unitary quantum groups are torsion-free. In the *third section*, we discuss torsion-freeness for general rigid tensor  $C^*$ -categories, and show that Cartesian and free products of torsion-free quantum groups are torsion-free.

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## 1 Torsion-free fusion rings

Let  $I$  be a pointed set with distinguished element  $\mathbb{1}$ , equipped with an involution  $\sigma : \alpha \mapsto \bar{\alpha}$  fixing  $\mathbb{1}$ . Consider the free  $\mathbb{Z}$ -module  $\mathbb{Z}_I$  with basis  $I$ , where we write addition by  $\oplus$ . Endow  $\mathbb{Z}_I$  with the  $\mathbb{Z}$ -linear functional  $\tau$  such that  $\tau(\alpha) = \delta_{\alpha, \mathbb{1}}$  and extend the involution on  $I$  to a  $\mathbb{Z}$ -linear involution  $\sigma : x \mapsto \bar{x}$ .

**Definition 1.1.** *We call  $I$ -based ring [16, 10, 22] any ring-structure  $\otimes$  on the abelian group  $\mathbb{Z}_I$ , with unit  $\mathbb{1}$  and such that, for all  $\alpha, \beta, \gamma \in I$ ,*

- $\overline{\alpha \otimes \beta} = \bar{\beta} \otimes \bar{\alpha}$ ,
- $\tau(\bar{\alpha} \otimes \beta) = \delta_{\alpha, \beta}$ ,
- $\tau(\alpha \otimes \beta \otimes \gamma) \in \mathbb{N}$ .

*We call fusion ring (over  $I$ ) an  $I$ -based ring  $(\mathbb{Z}_I, \otimes)$  which admits a dimension function, that is, a unital ring homomorphism  $d : (\mathbb{Z}_I, \otimes) \rightarrow \mathbb{R}$  such that, for all  $\alpha \in I$ ,*

- $d(\alpha) > 0$ ,
- $d(\alpha) = d(\bar{\alpha})$ .

For  $(\mathbb{Z}_I, \otimes)$  an  $I$ -based ring and  $x, y, z \in \mathbb{Z}_I$ , write

$$N_{yz}^x = \tau(y \otimes z \otimes \bar{x}). \quad (1)$$

Then it follows that, for  $x, y \in \mathbb{Z}_I$  fixed,  $N_{xy}^\alpha = 0$  for almost all  $\alpha \in I$ , and

$$x \otimes y = \oplus_\alpha N_{xy}^\alpha \alpha. \quad (2)$$

Note that by the second conditions,  $d(\alpha) \geq 1$  for  $\alpha \in I$  in a fusion ring with dimension function  $d$ .

When  $x, y \in \mathbb{Z}_I$ , we write  $x \leq y$  if  $y - x$  is an  $\mathbb{N}$ -linear combination of elements in  $I$ .

**Example 1.2.** Let  $\Gamma$  be a discrete group. Consider  $\Gamma$  with  $\mathbb{1}$  the unit of  $\Gamma$  and  $\bar{g} = g^{-1}$ . Then  $\mathbb{Z}_\Gamma$  is a fusion ring with  $g \otimes h = gh$  and dimension function  $d(g) = 1$ .

**Example 1.3.** Let  $\phi$  be the golden ratio. Then the ring  $\mathbb{Z}[\phi]$  becomes a fusion ring by giving it the  $\mathbb{Z}$ -basis  $\{1, \phi\}$  and trivial involution. Its dimension function is given by the inclusion  $d : \mathbb{Z}[\phi] \hookrightarrow \mathbb{R}$ .

**Example 1.4.** Consider  $\mathbb{N} = \{0, 1, \dots\}$  with distinguished element  $0$  and trivial involution. We can endow  $\mathbb{Z}_\mathbb{N}$  with the fusion ring structure

$$\mathbf{n} \otimes \mathbf{m} = \bigoplus_{k \in \{|n-m|, |n-m|+2, \dots, n+m\}} \mathbf{k}$$

and dimension function  $d(\mathbf{n}) = n+1$ . We will write this fusion ring as  $A(1)$ .

**Example 1.5.** Let  $F$  consist of words in the variables  $\pi_+, \pi_-$  with distinguished element the empty word and the unique involution which inverts word order and sends  $\pi_+$  to  $\pi_-$ . We can endow  $\mathbb{Z}_F$  with the fusion ring structure

$$w \otimes z = \sum_{\substack{\exists u \in F \\ xu=w, \bar{u}y=z}} xy$$

and the dimension function uniquely determined by  $d(\pi_+) = 2$ . We will write this fusion ring as  $A(2)$ .

**Example 1.6.** Let  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$  be fusion rings with respective dimension functions  $d_i$ . Write  $\odot$  for the tensor product over  $\mathbb{Z}$ . Then  $\mathbb{Z}_{I_1} \odot \mathbb{Z}_{I_2}$  is a fusion ring with basis  $\alpha \odot \beta$  for  $\alpha \in I_1, \beta \in I_2$ , with unit  $\mathbb{1} = \mathbb{1}_1 \odot \mathbb{1}_2$  and involution  $\overline{x \odot y} = \bar{x} \odot \bar{y}$ . It has a dimension function given by  $d(\alpha \odot \beta) = d_1(\alpha)d_2(\beta)$ . We call it the tensor product of  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$ .

**Example 1.7.** Let  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$  be fusion rings with respective dimension functions  $d_i$ . Let  $I_1 * I_2$  consist of alternating words, possibly empty, in  $I_1 \setminus \{\mathbb{1}_1\}$  and  $I_2 \setminus \{\mathbb{1}_2\}$ . We view  $I_1 * I_2$  as a pointed set with distinguished element  $\mathbb{1}$  the empty word, and endow it with the involution inverting order and acting as the involution of  $I_i$  on each letter. Then we can define a unique  $I_1 * I_2$ -based fusion ring with unit  $\mathbb{1}$  such that

$$\begin{aligned} w\alpha \otimes \beta z &= w\alpha\beta z, \quad \alpha \in I_i \setminus \{\mathbb{1}_i\}, \beta \notin I_i \sqcup \{\mathbb{1}_2\}, \\ w\alpha \otimes \beta z &= N_{\alpha, \beta}^{\mathbb{1}_i}(w \otimes z) \oplus \left( \bigoplus_{\gamma \in I_i \setminus \{\mathbb{1}_i\}} N_{\alpha, \beta}^{\gamma} w\gamma z \right), \quad \alpha, \beta \in I_i \setminus \{\mathbb{1}_i\}, \end{aligned}$$

and dimension function uniquely determined by  $d_{|I_i \setminus \{1_i\}} = d_i$ . We call it the free product of  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$

Evidently, the above two constructions can be performed with respect to an arbitrary number of components as well, viewed for example as inductive limits in the case of infinite index sets.

**Example 1.8.** Let  $(\mathbb{Z}_I, \otimes)$  be a fusion ring, and assume  $I' \subseteq I$  is a pointed subset which is invariant under the involution and for which  $N_{\alpha, \beta}^\gamma = 0$  for all  $\alpha, \beta \in I'$  and  $\gamma \in I \setminus I'$ . Then we obtain by restriction of  $\otimes$  a fusion ring  $(\mathbb{Z}_{I'}, \otimes)$ , which is called a fusion subring of  $(\mathbb{Z}_I, \otimes)$ .

For example, if  $(\mathbb{Z}_I, \otimes)$  is a fusion ring and  $\alpha \in I$ , one can consider the fusion subring generated by  $\alpha \in I$ , which is the smallest fusion subring of  $(\mathbb{Z}_I, \otimes)$  containing  $\alpha$ .

**Remark 1.9.** Clearly, an  $I$ -based ring is completely determined by  $(I, \mathbb{1}, \sigma)$  and a collection of numbers  $N_{\beta\gamma}^\alpha \in \mathbb{N}$  satisfying the following.

1.  $N_{\beta\gamma}^\alpha = N_{\gamma\alpha}^{\bar{\beta}} = N_{\alpha\beta}^{\bar{\gamma}} = N_{\bar{\gamma}\bar{\beta}}^{\bar{\alpha}}$  for all  $\alpha, \beta, \gamma$ .
2.  $N_{\mathbb{1}\beta}^\alpha = \delta_{\alpha, \beta}$  for all  $\alpha, \beta$ .
3. For all  $\beta, \gamma$  fixed,  $N_{\beta\gamma}^\alpha = 0$  for all but a finite number of  $\alpha$ .
4.  $\sum_\epsilon N_{\beta\gamma}^\epsilon N_{\alpha\epsilon}^\delta = \sum_\epsilon N_{\alpha\beta}^\epsilon N_{\epsilon\gamma}^\delta$  for all  $\alpha, \beta, \gamma, \delta$ .

**Definition 1.10.** Let  $J$  be a set. We call cofinite based module (over  $J$ ) for a based ring  $(\mathbb{Z}_I, \otimes)$  a (unital) left  $(\mathbb{Z}_I, \otimes)$ -module structure  $\otimes$  on  $\mathbb{Z}_J$ , together with a  $\mathbb{Z}_I$ -valued bilinear form  $\langle -, - \rangle$  satisfying, for all  $\alpha \in I$  and  $b, c \in J$ ,

- $\langle \alpha \otimes b, c \rangle = \alpha \otimes \langle b, c \rangle$ ,
- $\langle b, c \rangle = \overline{\langle c, b \rangle}$ ,
- $\tau(\langle b, c \rangle) = \delta_{b, c}$ ,
- $\tau(\alpha \otimes \langle b, c \rangle) \in \mathbb{N}$ .

If  $(\mathbb{Z}_I, \otimes)$  is a fusion ring, we will also call  $(\mathbb{Z}_J, \otimes)$  a cofinite fusion module. If then  $d$  is a dimension function on  $(\mathbb{Z}_I, \otimes)$ , we say a  $\mathbb{Z}$ -linear function  $d : \mathbb{Z}_J \rightarrow \mathbb{R}$  is a  $d$ -compatible dimension function if

- $d(a) > 0$  for all  $a \in J$ ,
- $d(\alpha \otimes a) = d(\alpha)d(a)$  for all  $\alpha \in I, a \in J$ .

Clearly there exists at least one dimension function on  $(\mathbb{Z}_J, \otimes)$  by defining

$$d(b) := d(\langle b, b_0 \rangle)$$

for some fixed element  $b_0 \in J$ .

For  $(\mathbb{Z}_J, \otimes)$  a cofinite based  $(\mathbb{Z}_I, \otimes)$ -module, we write

$$N_{xy}^z = \tau(x \otimes \langle y, z \rangle), \quad x \in \mathbb{Z}_I, y, z \in \mathbb{Z}_J. \quad (3)$$

Then for  $x \in \mathbb{Z}_I, y \in \mathbb{Z}_J, c \in J$ ,

$$x \otimes y = \oplus_c N_{xy}^c c. \quad (4)$$

Again, we write  $x \leq y$  if  $y - x$  is an  $\mathbb{N}$ -linear combination of elements in  $\mathbb{Z}_J$ .

**Example 1.11.** Let  $A = (\mathbb{Z}_I, \otimes)$  be a unital  $I$ -based ring. Then  $A$  is a cofinite  $I$ -based module over itself by left  $\otimes$ -multiplication and inner product

$$\langle \alpha, \beta \rangle = \alpha \otimes \bar{\beta}.$$

We will call this the standard  $I$ -based module.

**Remark 1.12.** Clearly, a cofinite  $J$ -based module is completely determined by numbers  $N_{\alpha b}^c \in \mathbb{N}$  satisfying the following.

1. For  $b, c \in J$  fixed,  $N_{\alpha b}^c = 0$  for all but a finite number of  $\alpha \in I$ .
2. For  $\alpha \in I, b \in J$  fixed,  $N_{\alpha b}^c = 0$  for all but a finite number of  $c \in J$ .
3.  $N_{1b}^c = \delta_{b,c}$  for all  $b, c \in J$ .
4.  $N_{\alpha b}^c = N_{\alpha c}^b$  for all  $b, c \in J$  and  $\alpha \in I$ .
5.  $\sum_e N_{\beta c}^e N_{\alpha e}^d = \sum_e N_{\alpha \beta}^e N_{e c}^d$  for all  $\alpha, \beta \in I, c, d \in J$ .

The associated module structure and inner product is determined by

$$\alpha \otimes b = \sum_c N_{\alpha b}^c c, \quad \langle b, c \rangle = \oplus_\alpha N_{\alpha b}^c \bar{\alpha}. \quad (5)$$

**Remark 1.13.** *If all conditions in Remark 1.12 are satisfied except possibly the first, then  $\mathbb{Z}_J$  is still a  $(\mathbb{Z}_I, \otimes)$ -module by (4). In this case, we call  $(\mathbb{Z}_J, \otimes)$  a (general) based  $(\mathbb{Z}_I, \otimes)$ -module.*

**Lemma 1.14.** *Let  $(\mathbb{Z}_J, \otimes)$  be a based  $(\mathbb{Z}_I, \otimes)$ -module. Then  $\forall \alpha \in I, b \in J$ , we have  $\alpha \otimes b \neq 0$ .*

*Proof.* If  $\alpha \otimes b = 0$ , then also  $\bar{\alpha} \otimes \alpha \otimes b = 0$ . But since  $1 \leq \bar{\alpha} \otimes \alpha$ , it follows that  $1 = N_{1,b}^b \leq N_{\bar{\alpha} \otimes \alpha, b}^b = 0$ , a contradiction.  $\square$

**Definition 1.15.** *We call a  $J$ -based  $(\mathbb{Z}_I, \otimes)$ -module connected if for any  $b, c \in J$ , there exists  $\alpha \in I$  with  $N_{\alpha b}^c \neq 0$ .*

By the fourth identity in Remark 1.12, it is enough to require the above condition for a single fixed  $b$ . This immediately implies that any (cofinite) based  $(\mathbb{Z}_I, \otimes)$ -module is a (possibly infinite) direct sum of connected (cofinite) based modules.

**Lemma 1.16.** *Let  $(\mathbb{Z}_J, \otimes)$  be a connected based  $(\mathbb{Z}_I, \otimes)$ -module. Assume that  $b, c \in J$  are such that  $N_{\alpha b}^c = 0$  for all but a finite number of  $\alpha$ . Then  $(\mathbb{Z}_J, \otimes)$  is cofinite.*

*Proof.* For  $\beta \in I$  fixed,  $N_{\alpha, \beta \otimes b}^c = N_{\alpha \otimes \beta, b}^c = 0$  for all but a finite number of  $\alpha$ . Take now  $d \in J$  arbitrary. Then by connectedness, we can find  $\beta \in I$  with  $d \leq \beta \otimes b$ . As  $N_{\alpha d}^c \leq N_{\alpha, \beta \otimes b}^c$ , it follows that  $N_{\alpha d}^c = 0$  for all but a finite number of  $\alpha$ . Since  $N_{\alpha d}^c = N_{\alpha c}^d$ , we can replace also  $c$  by an arbitrary element in  $J$  to conclude that  $(\mathbb{Z}_J, \otimes)$  is cofinite.  $\square$

To elements of a fusion ring acting on a based module, one can associate combinatorial data in the following way.

**Notation 1.17.** *If  $(\mathbb{Z}_J, \otimes)$  is a based  $(\mathbb{Z}_I, \otimes)$ -module, we write  $M_J(\alpha)$  for the matrix  $M_J(\alpha)_{b,c} = N_{\alpha b}^c$ . We write  $\Gamma_J(\alpha)$  for the associated (oriented) graph having  $M_J(\alpha)$  as its adjacency matrix, so that there are  $M_J(\alpha)_{b,c}$  arrows from  $b$  to  $c$ .*

Note that on each row/column of  $M_J(\alpha)$ , there are only finitely many non-zero entries, which are then positive integers. It follows that we can apply  $M(\alpha)$  to any vector  $v \in \mathbb{C}^J$ . Note also that  $M(\alpha \otimes \beta) = M(\alpha)M(\beta)$  and  $M(\bar{\alpha}) = M(\alpha)^*$ .

If  $(\mathbb{Z}_J, \otimes)$  is a fusion module over the fusion ring  $(\mathbb{Z}_I, \otimes)$ , with a pair of compatible dimension functions  $d$ , we can view  $d$  as a vector  $D \in \mathbb{C}^J$  with  $D_b = d(b)$ . It follows that for any  $\alpha \in I$ ,

$$(M_J(\alpha)D)_c = \sum_b M_J(\alpha)_{c,b}d(b) = d(\oplus_b N_{\alpha c}^b b) = d(\alpha \otimes c) = d(\alpha)D_c. \quad (6)$$

As  $D$  is also an eigenvector at this eigenvalue for  $M_J(\overline{\alpha})$ , we obtain by the Schur test that  $M_J(\alpha)$  is a bounded matrix on  $\ell^2(J)$  with  $\|M_J(\alpha)\| \leq d(\alpha)$ .

The following is our main definition.

**Definition 1.18.** *We call a fusion ring torsion-free if any non-zero connected cofinite based module is isomorphic to the standard based module.*

In the above definition, an isomorphism of based modules is assumed to take basis elements to basis elements.

**Proposition 1.19.** *Let  $\Gamma$  be a discrete group. Then  $\Gamma$  is torsion-free if and only if  $(\mathbb{Z}_\Gamma, \otimes)$  is torsion-free.*

*Proof.* This follows immediately from the fact that a non-zero cofinite  $J$ -based  $(\mathbb{Z}_\Gamma, \otimes)$ -module is determined by an action  $\Gamma \curvearrowright J$  with finite stabilizers.  $\square$

**Proposition 1.20.** *Assume  $(\mathbb{Z}_I, \otimes)$  is a torsion-free fusion ring with dimension function  $d$ . Then  $\Gamma = \{g \in I \mid d(g) = 1\}$  is torsion-free.*

*Proof.* Note that  $(\Gamma, \otimes)$  is a discrete group with a natural right action on  $\mathbb{Z}_I$  by  $\otimes$ . If  $\Gamma$  is not torsion-free, let  $H$  be a non-trivial finite subgroup of  $\Gamma$ . Then  $(\mathbb{Z}_{I/H}, \otimes)$  is a non-zero connected cofinite fusion module for  $(\mathbb{Z}_I, \otimes)$  in a natural way, hence isomorphic to  $(\mathbb{Z}_I, \otimes)$  as a based  $(\mathbb{Z}_I, \otimes)$ -module. Since this isomorphism must preserve basis elements of minimal dimension, it sends  $\Gamma/H$  to  $\Gamma$  in a  $\Gamma$ -equivariant way, which gives a contradiction.  $\square$

**Proposition 1.21.** *If  $(\mathbb{Z}_I, \otimes)$  is a fusion ring with finite  $I$  and integer-valued dimension function, then  $(\mathbb{Z}_I, \otimes)$  is torsion-free if and only if  $I$  is a singleton.*

*Proof.* Assume  $(\mathbb{Z}_I, \otimes)$  is a fusion ring with integer-valued dimension function  $d$ . Then we can endow  $\mathbb{Z} = \mathbb{Z}_{\{\bullet\}}$  with the non-zero cofinite  $\{\bullet\}$ -based  $(\mathbb{Z}_I, \otimes)$ -module structure  $\alpha \otimes \bullet = d(\alpha)\bullet$ . Hence if  $(\mathbb{Z}_I, \otimes)$  is torsion-free, we must have  $I \cong \{\bullet\}$ .  $\square$



**Proposition 1.22.** *The fusion ring  $(\mathbb{Z}[\phi], \otimes)$  is torsion-free.*

*Proof.* Any connected fusion module for  $(\mathbb{Z}[\phi], \otimes)$  is completely determined by  $M = M_J(\phi)$ , which is a symmetric, positive integer-valued matrix with norm  $\phi$ , since  $M^2 = M + 1$ . As  $M$  determines a connected graph, it follows by the classification of graphs with small norms (see e.g. [11]) that necessarily (up to permutation)  $M$  is the tadpole  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ , that is,  $M$  determines the standard module.  $\square$

The above result shows that finite fusion rings can be non-trivial yet torsion-free.

We now aim to show that the fusion rings  $A(1)$  and  $A(2)$  in Examples 1.4 and 1.5 are torsion-free.

**Proposition 1.23.** *The fusion ring  $A(1)$  is torsion-free.*

*Proof.* Let  $(\mathbb{Z}_J, \otimes)$  be a non-zero cofinite fusion  $A(1)$ -module with chosen compatible dimension function  $d(b) = d(\langle b, b_0 \rangle)$  with respect to some fixed  $b_0 \in J$ . Write  $M = M_J(\mathbf{1})$ . Let  $\Lambda$  be the unoriented graph associated with  $M$ . Then  $\|\Lambda\| = \|M\| = 2$ , and  $\Lambda$  is connected by connectedness of  $(\mathbb{Z}_J, \otimes)$  and the fact that  $\mathbf{1}$  is generating for  $A(1)$ . Hence  $\Lambda$  is an extended Dynkin diagram, possibly with loops. For convenience, we refer to the list of these graphs in [23, Appendix].

By immediate inspection, all extended Dynkin diagrams except for  $A_\infty$  have a Frobenius-Perron eigenvector at eigenvalue 2 which is uniformly bounded. Hence, suppose that  $\Lambda$  is not  $A_\infty$ . By unicity of the Frobenius-Perron eigenvector up to a scalar multiple, it follows that  $d$  must be uniformly bounded on  $J$ . However, for  $b \in J$  we have

$$d(b) = d(\langle b, b_0 \rangle) = \sum_{n \geq 0} N_{\mathbf{n}, b}^{b_0} d(\mathbf{n}) = \sum_{n \geq 0} (n + 1) N_{\mathbf{n}, b}^{b_0}.$$

Hence, we infer that there exists  $m \geq 0$  such that  $N_{\mathbf{n}, b}^{b_0} = 0$  for all  $b \in J$  and all  $n \geq m$ . It follows that  $M = \langle \mathbb{Z}_J, b_0 \rangle$  is a non-trivial finite rank submodule of  $A(1)$ . Clearly, this is impossible.

It follows that  $\Lambda$  is an  $A_\infty$ -graph, and it is then immediate to conclude that  $(\mathbb{Z}_J, \otimes)$  is isomorphic to the standard based  $A(1)$ -module, since the action of  $\mathbf{1}$  determines the complete module.  $\square$

**Theorem 1.24.** *The fusion ring  $A(2)$  is torsion-free.*

*Proof.* Let  $(\mathbb{Z}_J, \otimes, d)$  be a non-zero cofinite fusion  $A(2)$ -module with chosen compatible dimension function  $d$ , which we may assume integer-valued. Write  $M(\pm) = M_J(\pi_\pm)$ ,  $\Gamma(\pm) = \Gamma_J(\pi_\pm)$ . It is enough to prove that  $\Gamma(+)$  is  $\Gamma_F(\pi_+)$ , the graph associated to  $\pi_+$  for the standard module.

Since  $\pi_+^{\otimes n} = \pi_+^n$ , it follows that  $N_{\pi_+, b}^b \neq 0$  for some  $b \in J$  implies  $N_{\pi_+, b}^b \neq 0$  for all  $n \in \mathbb{N}$ . By cofiniteness of  $(\mathbb{Z}_J, \otimes)$ , it follows that  $N_{\pi_+, b}^b = 0$  for all  $b \in J$ , and  $\Gamma(+)$  has no loops.

Write  $\tilde{J} = J \times \{-, +\}$ , whose elements we will write as  $b_\pm$ . Let  $\tilde{\Lambda}$  be the unoriented graph with vertex set  $\tilde{J}$  and  $M(\pm)_{b,c}$  edges between  $b_\pm$  and  $c_\mp$ , and no other edges. Endow  $\mathbb{Z}[\tilde{J}]$  with the unique based  $A(1)$ -module structure such that  $N_{\mathbf{1}, c_\mu}^{d_\nu} = \tilde{M}_{c_\mu, d_\nu}$ , with  $\tilde{M}$  the (self-adjoint) adjacency matrix of  $\tilde{\Lambda}$ .

As  $(M(\pi_{\mu_n}) \dots M(\pi_\mu) M(\pi_{-\mu}) M(\pi_\mu))_{b,c}$  counts the number of paths of length  $n$  from  $c$  to  $b$  in  $\Gamma(\pi)$  which alternate orientation and start in the direction  $\pi_\mu$ , it follows that this number equals  $(\tilde{M}^n)_{b_{\mu_n}, c_\mu}$ . But by induction, one easily verifies by use of the fusion rules that there exist  $P_k \in \mathbb{Z}$  such that

$$\pi_{\mu_n} \dots \pi_\mu \pi_{-\mu} \pi_\mu = \sum_{k=0}^n P_k \pi_{\mu_k} \otimes \dots \otimes \pi_\mu \otimes \pi_{-\mu} \otimes \pi_\mu, \quad \mathbf{n} = \sum_{k=0}^n P_k \mathbf{1}^{\otimes k}.$$

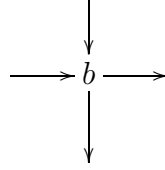
We conclude that

$$N_{\mathbf{n}, b_{\mu_n}}^{c_\mu} = N_{\pi_{\mu_n} \dots \pi_\mu \pi_{-\mu} \pi_\mu, b}^c, \quad b, c \in J.$$

In particular,  $N_{\mathbf{n}, b_{\mu_n}}^{c_\mu} = 0$  for  $n$  large. We deduce that  $\mathbb{Z}[\tilde{J}]$  is a cofinite based  $A(1)$ -module, and so  $\tilde{\Lambda}$  is a disjoint union of  $A_\infty$ -graphs.

It follows from the above that  $\Gamma(+)$  can not contain any double arrows  $\Rightarrow$  or  $\Leftarrow$  between two vertices, and that for any  $b \in J$ , there are at most 2 edges incoming and at most 2 edges outgoing from any vertex  $b$ . In fact, the

situation



can not appear either. Indeed, this would imply

$$b \oplus (\pi\rho \otimes b) = \pi \otimes \rho \otimes b = 2b \oplus \dots,$$

and hence  $N_{\pi\rho,b}^b = 1$ . Also  $N_{\rho\pi,b}^b = 1$ . But  $\pi\rho^2\pi^2\rho^2\dots = \pi\rho \otimes \rho\pi \otimes \pi\rho \otimes \dots$ , so  $N_{\alpha,b}^b$  is nonzero for all alternating products  $\alpha$  of  $\pi\rho$  and  $\rho\pi$ , contradicting the cofiniteness condition.

Consider now  $\Lambda$ , the unoriented graph underlying  $\Gamma(+)$ . (There is no ambiguity in this definition as  $\Gamma(+)$  does not have loops or multiple edges.) We will show that  $\Lambda$  is in fact a tree.

Let  $b_0$  be a vertex of minimal dimension. Then in  $\Gamma(+)$  there can not be two edges leaving or two edges arriving in  $b_0$ , as one of their endpoints would have dimension strictly smaller than  $d(b_0)$ . As there has to be at least one edge arriving and one edge leaving, by Lemma 1.14, the situation around  $b_0$  is as follows:  $b \leftarrow b_0 \rightarrow c$ , with  $d(b) = d(c) = 2d(b_0)$ .

Let  $b_0 \xrightarrow{e_1} b_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} b_n$  be a path in  $\Lambda$ . We claim that  $d(b_i) < d(b_{i+1})$ . We have already shown that  $d(b_0) < d(b_1)$ . Suppose then that  $d(b_{i-1}) < d(b_i)$  for  $i \geq 1$ . By this dimensional assumption, there must be another edge with vertex  $b_i$  of the same orientation as  $e_i$ . As there also has to be an edge to  $b_i$  of the opposite orientation, we must be in one of the following situations:

$$\begin{array}{cc}
 \begin{array}{c} \downarrow \\ b_{i-1} \leftarrow b_i \longrightarrow b_{i+1} \end{array} & \begin{array}{c} \uparrow \\ b_{i-1} \leftarrow b_i \leftarrow b_{i+1} \end{array} \\
 \begin{array}{c} \uparrow \\ b_{i-1} \longrightarrow b_i \longleftarrow b_{i+1} \end{array} & \begin{array}{c} \downarrow \\ b_{i-1} \longrightarrow b_i \longrightarrow b_{i+1} \end{array}
 \end{array} \tag{7}$$

In each of these situations,  $d(b_{i+1}) > d(b_i)$ . It now follows immediately that  $\Lambda$  must be a tree.

From the above, we can already conclude that  $J \cong F$ , and that  $\Lambda$  coincides with the unoriented graph underlying  $\Gamma_F(\pi_+)$ . Let us show that in fact  $\Gamma(+) = \Gamma_F(\pi_+)$ . Indeed, let again  $b_0$  be a vertex of minimal dimension, viewed as the root of the tree  $\Gamma(+)$ . Then from level 0 to level 1, we have the situation



By induction, it is clear from the only possible choices (7) that this pattern repeats itself from level  $n$  to level  $n + 1$  at each vertex. This concludes the proof.  $\square$

We now prove that free products of torsion-free fusion rings are torsion-free.

**Theorem 1.25.** *Let  $A_s = (\mathbb{Z}_{I_s}, \otimes)$  be torsion-free fusion rings for  $s \in S$ . Then their free product  $A = \ast_{s \in S} A_s$  is again torsion-free.*

*Proof.* Let  $B = (\mathbb{Z}_J, \otimes)$  be a non-zero connected cofinite based fusion module for  $A$  with  $d = \ast_{s \in S} d_s$ -compatible dimension function  $d(b) = d(\langle b, b_0 \rangle)$  for some  $b_0 \in J$ . For each  $b \in J$ , let  $B_s^b$  with basis  $J_s^b$  be the connected cofinite based  $A_s$ -submodule of  $B$  spanned by  $b$ . By assumption,  $B_s^b \cong A_s$  as  $A_s$ -based module for any  $b \in J$ .

**Claim:** There exists  $e \in J$  such that  $\beta \otimes \alpha \otimes e \in J$  for each  $\beta \in \sqcup_s I_s$  and each  $\alpha \in I$  with  $d(\alpha) = 1$ .

Indeed, assume that  $b \in J$  and  $\beta \otimes b \notin J$  for some  $\beta \in I_s$ . Choose  $c \in B_s^b$  such that  $\gamma \otimes c$  is irreducible for each  $\beta \in I_s$ . With  $\gamma$  such that  $\gamma \otimes c = b$ , we must necessarily have that  $d_s(\gamma) > 1$ . In fact, since  $d_s(\gamma)$  is bounded below by the norm of a matrix of positive integers, we must then have  $d_s(\gamma) \geq \sqrt{2}$ , and so  $d(b) \geq \sqrt{2}d(c)$ . Replacing  $b$  by  $\alpha \otimes c$  for  $\alpha \in I$  with  $d(\alpha) = 1$ , we can iterate this argument, which must stop at some point as  $d(a) \geq 1$  for all  $a \in J$ . Hence there must exist  $e$  as in the claim.

Fix now  $e \in J$  satisfying the above property. Then by its definition, it satisfies the 0th step in the following induction argument.

**Claim:** For any reduced word  $\alpha = \alpha_0 \alpha_1 \dots \alpha_n$  where  $\alpha_i \in I_{s(i)}$ , we have  $\alpha \otimes e \in J$ . Moreover for any  $s \neq s(0)$ , we have an isomorphism  $\varphi : B_s^{\alpha \otimes e} \rightarrow A_s$  of based  $A_s$ -modules such that  $\varphi(\alpha \otimes e) = \mathbb{1}_s$ .

Indeed, take any  $\alpha$  as above and assume the claim holds for  $\alpha^k = \alpha_k \alpha_{k+1} \dots \alpha_n$  for each  $k \geq 1$ . Thanks to the isomorphism  $\varphi' : B_{s(0)}^{\alpha^1 \otimes e} \rightarrow A_{s(0)}$  as above, we get  $\varphi'(\alpha \otimes e) = \alpha_0 \in J_{s(0)}$ , and hence  $\alpha \otimes e \in J_{s(0)}^{\alpha^1 \otimes e} \subseteq J$ .

To show that  $\varphi : B_s^{\alpha \otimes e} \rightarrow A_s$  exists, note that there necessarily exists an isomorphism  $\varphi_0 : B_s^{\alpha \otimes e} \rightarrow A_s$  of based  $A_s$ -modules, by the first part of the proof. It hence suffices to show

$$d_s(\varphi_0(\alpha \otimes e)) = 1.$$

Assume not. Then there exists  $\mathbb{1}_s \neq \beta \in I_s$  such that  $N_{\beta, \alpha \otimes e}^{\alpha \otimes e}$  is nonzero. Moreover,  $d(\alpha) \neq 1$  by our assumption and the defining property of  $e$ . Pick a minimal  $k$  such that  $d_{s(k)}(\alpha_k) \neq 1$ . Again we have  $\mathbb{1}_{s(k)} \neq \gamma_0 \in A_{s(k)}$  such that  $N_{\gamma_0, \alpha^k \otimes e}^{\alpha^k \otimes e}$  is nonzero, hence for any alternating product  $\delta$  of  $\beta$  and  $\gamma := \alpha_0 \alpha_1 \dots \alpha_{k-1} \gamma_0 \overline{\alpha_{k-1}} \dots \overline{\alpha_0}$ , we have  $N_{\delta, \alpha \otimes e}^{\alpha \otimes e} \neq 0$ , contradicting cofiniteness. Therefore we have proven the claim.

The claim shows that  $\alpha \otimes e \in J$  for all  $\alpha \in I$ . We now show that these are mutually distinct. We only need to show  $N_{\alpha, e}^e = 0$  for any  $\alpha \neq \mathbb{1}$ . But assume  $N_{\alpha, e}^e \neq 0$ . Since  $\alpha \otimes e \in J$ , we get  $e = \alpha \otimes e$ . In particular,  $d(\alpha) = 1$ . Hence  $\alpha \in \Gamma = *_s \Gamma_s$  with  $\Gamma_s \subseteq I_s$  the group of elements in  $I_s$  with dimension 1. Since the  $\Gamma_s$  are torsion-free by Proposition 1.20, also  $*_s \Gamma_s$  is torsion-free. As the  $\mathbb{Z}_\Gamma$ -submodule generated by  $e$  is cofinite, we deduce that  $\alpha = \mathbb{1}$ .

It now follows from the above that

$$A \rightarrow B : \alpha \mapsto \alpha \otimes e$$

is a based module isomorphism, finishing the proof.  $\square$

By contrast, torsion-freeness is in general not preserved by tensor products. Indeed, if  $(\mathbb{Z}_I, \otimes)$  is a non-trivial torsion-free fusion ring with  $I$  finite, then  $\mathbb{Z}_I$  can be made into a non-standard (connected, cofinite) fusion module for  $(\mathbb{Z}_I \odot \mathbb{Z}_I, \otimes)$  by

$$(\alpha \odot \beta) \otimes \gamma = \alpha \otimes \gamma \otimes \overline{\beta}.$$

However, this is in a sense the only thing which can go wrong.

**Theorem 1.26.** *Let  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$  be torsion-free fusion rings, and assume  $(\mathbb{Z}_{I_1} \odot \mathbb{Z}_{I_2}, \otimes)$  is not torsion-free. Then  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$  have non-trivial isomorphic finite fusion subrings.*

*Proof.* Let  $(\mathbb{Z}_J, \otimes)$  be a non-zero, non-standard connected cofinite fusion module for  $(\mathbb{Z}_I, \otimes) = (\mathbb{Z}_{I_1} \odot \mathbb{Z}_{I_2}, \otimes)$ . As in the proof of Theorem 1.25, we can find  $e \in J$  such that  $\alpha \otimes e$  is irreducible for each  $\alpha \in I_1 \sqcup I_2$ , and such that all  $\alpha \otimes e$  are mutually distinct for  $\alpha \in I_1$  (resp.  $\alpha \in I_2$ ).

We claim that  $\langle e, e \rangle \neq 1$ . Indeed, by connectedness we have  $\langle c, b \rangle \neq 0$  for each  $c, b \in J$ . Hence, if  $\langle e, e \rangle$  were 1, then since  $\alpha = \langle \alpha \otimes e, e \rangle$  for each  $\alpha \in I$ , it would follow that  $\alpha \otimes e \in J$  for each  $\alpha \in I$ . Similarly, since  $\langle \alpha_1 \otimes e, \alpha_2 \otimes e \rangle = \alpha_1 \otimes \bar{\alpha}_2$ , it would follow that the  $\alpha \otimes e$  are distinct for distinct  $\alpha \in I$ . Hence  $(\mathbb{Z}_J, \otimes)$  would be the standard  $(\mathbb{Z}_I, \otimes)$ -module, in contradiction with the assumption.

There hence exist an  $\alpha_1 \in I_1$  and an  $\alpha_2 \in I_2$ , not both the unit element, such that  $\langle e, e \rangle = (\alpha_1 \otimes \bar{\alpha}_2) \oplus \dots$ , where we identify  $(\mathbb{Z}_{I_1}, \otimes)$  and  $(\mathbb{Z}_{I_2}, \otimes)$  as (commuting) fusion subrings of  $(\mathbb{Z}_I, \otimes)$ . Hence

$$0 \neq N_{\alpha_1 \otimes \alpha_2, e}^e = N_{\bar{\alpha}_1, \alpha_2 \otimes e}^e = N_{\alpha_1, e}^{\alpha_2 \otimes e} = N_{1, \alpha_1 \otimes e}^{\alpha_2 \otimes e},$$

and we deduce  $\alpha_1 \otimes e = \alpha_2 \otimes e$ . Since  $\overline{\langle e, e \rangle} = \langle e, e \rangle$ , we must have as well  $\bar{\alpha}_1 \otimes e = \bar{\alpha}_2 \otimes e$ .

Let  $(\mathbb{Z}_{I'_i}, \otimes)$  be the fusion subring of  $(\mathbb{Z}_{I_i}, \otimes)$  generated by  $\alpha_i$ . For  $w = w_1 \dots w_n$  a word in  $\{+, -\}$ , let  $w(\alpha_i) = \alpha_i^{w_1} \otimes \dots \otimes \alpha_i^{w_n}$ , where  $\alpha_i^+ = \alpha_i$  and  $\alpha_i^- = \bar{\alpha}_i$ . From the above, and from the fact that elements in  $I_1$  and  $I_2$  commute, we have that  $w(\alpha_2) \otimes e = w^o(\alpha_1) \otimes e$ , where  $w^o = w_n \dots w_1$ . It follows that there exists an identification

$$I'_1 \cong I'_2, \quad \beta_1 \leftrightarrow \beta_2$$

such that  $\beta_1 \otimes e = \beta_2 \otimes e$ .

Now for each  $\beta_1 \in I'_1$ , we have

$$N_{\beta_1 \otimes \beta_2, e}^e = N_{\beta_1, \beta_2 \otimes e}^e = N_{\beta_1, \beta_1 \otimes e}^e = N_{\beta_1, e}^{\beta_1 \otimes e} = N_{1, \beta_1 \otimes e}^{\beta_1 \otimes e} = 1.$$

By cofiniteness, we deduce that  $I'_1$  and  $I'_2$  are finite.

It follows that the  $(\mathbb{Z}_{I'_i}, \otimes)$  are finite fusion subrings  $(\mathbb{Z}_{I_i}, \otimes)$ . If  $d(\alpha_1) = 1$ , then we must have  $\alpha_1 = \mathbb{1}_1$  by Proposition 1.20. Since  $d(\alpha_1) = d(\alpha_2)$ , and since either  $\alpha_1$  or  $\alpha_2$  is not a unit, it thus follows that neither of them is a unit. Hence both  $(\mathbb{Z}_{I'_i}, \otimes)$  are non-trivial finite fusion subrings. In fact, it is easily seen that  $\beta_1 \mapsto \overline{\beta_2}$  is an isomorphism  $(\mathbb{Z}_{I_1}, \otimes) \rightarrow (\mathbb{Z}_{I_2}, \otimes)$ .  $\square$

Torsion-freeness does also not automatically pass to fusion subrings. Consider for example the fusion subring of  $A(1)$  generated by **2** and the cofinite based module for it generated by **1** inside  $A(1)$ . We can say something more however under extra assumptions. The following definition is a straightforward generalisation of [25, Definition 4.1], in its equivalent characterisation given by [25, Lemma 4.2.d)].

**Definition 1.27.** *Let  $(\mathbb{Z}_I, \otimes)$  be a fusion ring with fusion subring  $(\mathbb{Z}_{I'}, \otimes)$ . We call  $(\mathbb{Z}_{I'}, \otimes)$  a divisible fusion subring of  $(\mathbb{Z}_I, \otimes)$  if  $(\mathbb{Z}_I, \otimes) \cong \oplus(\mathbb{Z}_{I'}, \otimes)$  as based right  $(\mathbb{Z}_{I'}, \otimes)$ -modules.*

Using the involution, we see that we may replace right by left in the above definition.

**Proposition 1.28.** *A divisible fusion subring  $(\mathbb{Z}_{I'}, \otimes)$  of a torsion-free fusion ring  $(\mathbb{Z}_I, \otimes)$  is again torsion-free.*

*Proof.* Assume  $(\mathbb{Z}_{J'}, \otimes)$  is a non-zero cofinite connected fusion  $(\mathbb{Z}_{I'}, \otimes)$ -module. Choose an identification  $(\mathbb{Z}_I, \otimes) \cong \oplus(\mathbb{Z}_{I'}, \otimes)$  of based right  $\mathbb{Z}_{I'}$ -modules, and let  $I_0 = \{\alpha_0\}$  be the elements in  $I$  corresponding to the units of the components  $\mathbb{Z}_{I'}$ . Then, by assumption,  $\mathbb{Z}_J = \mathbb{Z}_I \otimes_{\mathbb{Z}_{I'}} \mathbb{Z}_{J'}$  is a based  $\mathbb{Z}_I$ -module with basis  $J = \{\alpha_0 \otimes b \mid \alpha_0 \in I_0, b \in J'\}$ , and it is cofinite by the inner product

$$\langle \alpha \otimes b, \gamma \otimes d \rangle = \alpha \otimes \langle b, d \rangle \otimes \overline{\gamma}.$$

It is clearly also connected. Hence  $(\mathbb{Z}_J, \otimes) \cong (\mathbb{Z}_I, \otimes)$  as a based  $(\mathbb{Z}_I, \otimes)$ -module.

Let  $\alpha_0 \otimes b_0$  be the element corresponding to  $\mathbb{1} \in \mathbb{Z}_I$ . Then  $\alpha \otimes \alpha_0 \otimes b_0 \in J$  for each  $\alpha \in I$ . It follows in particular that  $\alpha \otimes \alpha_0 \in I$  for each  $\alpha \in I$ , and so  $\overline{\alpha_0} \otimes \alpha_0 = \mathbb{1}$ . We may thus assume  $\alpha_0 = \mathbb{1}$ , and we find  $\mathbb{Z}_{J'} \cong \mathbb{1} \otimes_{\mathbb{Z}_{I'}} \mathbb{Z}_{J'} \cong \mathbb{Z}_{I'}$  as based  $(\mathbb{Z}_{I'}, \otimes)$ -modules.  $\square$

**Remark 1.29.** *It follows from [25, Proposition 4.3] that  $A(2)$  is a divisible fusion subring of  $A(1) * \mathbb{Z}_{\mathbb{Z}}$ , so we obtain an alternative proof of Theorem 1.24 by combining Theorem 1.25, Proposition 1.23, Proposition 1.19 and Proposition 1.28.*

## 2 Strong torsion-freeness for discrete quantum groups

Let  $\Gamma$  be a discrete quantum group, by which we will, for the sake of convenience, understand a Hopf  $*$ -algebra  $(\mathbb{C}[\Gamma], \Delta)$  with invariant (positive) state  $\varphi : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}$ . In particular,  $(\mathbb{C}[\Gamma], \Delta)$  is a cosemisimple Hopf algebra. Endow  $\mathbb{C}[\Gamma]$  with the pre-Hilbert structure  $\langle x, y \rangle = \varphi(x^*y)$ .

By *corepresentation*  $X$  of  $(\mathbb{C}[\Gamma], \Delta)$  we will always understand *finite-dimensional* corepresentation, that is, a finite-dimensional vector space  $V$  together with a linear map  $\delta : V \rightarrow V \otimes \mathbb{C}[\Gamma]$  such that

$$(\text{id} \otimes \Delta)\delta = (\delta \otimes \text{id})\delta, \quad (\text{id} \otimes \varepsilon)\delta = \text{id}.$$

We call a corepresentation  $V = (V, \delta)$  *unitary* if  $V$  is equipped with a Hilbert space structure for which  $\delta$  is *isometric*, that is

$$\delta(\xi)^* \delta(\eta) = \langle \xi, \eta \rangle 1 \in \mathbb{C}[\Gamma]$$

for all  $\xi, \eta \in V$ , where we write  $\xi^* \eta = \langle \xi, \eta \rangle$ .

The unitary corepresentations form a rigid tensor  $C^*$ -category [15, 19] with unit the trivial corepresentation  $\mathbb{1} = \mathbb{C}$ . We will write the morphism spaces of  $\Gamma$ -equivariant linear maps as  $\text{Mor}^\Gamma(V, W)$ .

**Definition 2.1.** *Let  $\Gamma$  be a discrete quantum group. We associate to  $\Gamma$  the fusion ring  $\text{Fus}(\Gamma)$  with basis the set  $I = \{[V]\}$  of equivalence classes of irreducible (unitary) corepresentations, with unit  $\mathbb{1} = [\mathbb{1}]$  and involution  $\overline{[V]} = [\overline{V}]$ , where  $\overline{V}$  is the dual of  $V$ , and fusion rules and dimension function*

$$N_{[V], [W]}^{[Z]} = \dim(\text{Mor}^\Gamma(Z, V \otimes W)), \quad d([V]) = \dim(V).$$

See for example [4], [19, Section 2.7] for more information.



If  $(A, \alpha)$  is a corepresentation which also has the structure of a  $C^*$ -algebra for which  $\alpha$  is a unital  $*$ -homomorphism, we call  $(A, \alpha)$  a *coaction* of  $(\mathbb{C}[\Gamma], \Delta)$ . The following lemma is closely related to the discussion on  $\delta$ -forms in [5].

**Lemma 2.2.** *If  $(A, \alpha)$  is a coaction, there exists a positive functional  $\varphi_A$  on  $A$  such that  $\langle a, b \rangle = \varphi_A(a^*b)$  turns  $(A, \alpha)$  into a unitary corepresentation for which the multiplication map*

$$m : A \otimes A \rightarrow A, \quad a \otimes b \mapsto ab$$

*is a coisometry.*

*Proof.* Let  $\omega$  be a positive faithful state on  $A$ , and let

$$\varphi'_A(a) = (\omega \otimes \varphi)\alpha(a).$$

From the invariance of  $\varphi$ , one deduces that  $\varphi'_A$  is invariant, that is,

$$(\varphi'_A \otimes \text{id})(\alpha(a)) = \varphi'_A(a)1 \in \mathbb{C}[\Gamma].$$

Moreover, as we chose  $\omega$  faithful, and as  $\varphi$  is faithful, also  $\varphi'_A$  is faithful. It is then easy to see that  $\langle a, b \rangle = \varphi'_A(a^*b)$  turns  $(A, \alpha)$  into a unitary corepresentation.

There now exists a positive, invertible, central element  $T \in A$  such that, with respect to the above scalar product,  $mm^*(a) = Ta$ , namely the *index* of  $\varphi'_A : A \rightarrow \mathbb{C}$  [29, Definition 1.2.2., Proposition 1.2.8]. As by definition  $m \in \text{Mor}^\Gamma(A \otimes A, A)$ , it follows that  $\alpha(T) = T \otimes 1$ . Hence  $\varphi_A(a) = \varphi'_A(T^{-1}a)$  satisfies the requirements of the lemma.  $\square$

We will in the following always consider a coaction with an invariant faithful positive functional as above.

For example, if  $(V, \delta)$  is a unitary corepresentation, then  $B(V)$  carries the *adjoint coaction*

$$B(V) \rightarrow B(V) \otimes \mathbb{C}[\Gamma], \quad \xi \eta^* \mapsto \delta(\xi)\delta(\eta)^*.$$

**Definition 2.3** ([1]). *Let  $(A, \alpha)$  be a coaction of  $(\mathbb{C}[\Gamma], \Delta)$ . An equivariant (right) Hilbert  $A$ -module is a corepresentation  $(\mathcal{E}, \delta)$  where  $\mathcal{E}$  is equipped with a right Hilbert  $A$ -module structure for which*

$$\delta(\xi a) = \delta(\xi)\alpha(a), \quad \langle \delta(\xi), \delta(\eta) \rangle_{A \otimes \mathbb{C}[\Gamma]} = \alpha(\langle \xi, \eta \rangle_A).$$

One can turn  $(\mathcal{E}, \delta)$  into a unitary corepresentation by the inner product  $\langle \xi, \eta \rangle = \varphi_A(\langle \xi, \eta \rangle_A)$ . Since any Hilbert  $A$ -module is a direct summand of some  $\mathbb{C}^n \otimes A$ , it follows that the module maps  $\mathcal{E} \otimes A \rightarrow \mathcal{E}$  are coisometries. Moreover, since the space of equivariant  $A$ -module maps is closed under the adjoint operation, any equivariant Hilbert module decomposes as a direct sum of irreducible equivariant Hilbert modules.

If  $V$  is a unitary corepresentation of  $\Gamma$ , then  $V \otimes \mathcal{E}$  is again an equivariant Hilbert  $A$ -module by the tensor product corepresentation and the  $A$ -module action only on  $\mathcal{E}$ . We hence obtain from  $(A, \alpha)$  a fusion module for  $\text{Fus}(\Gamma)$  in the following way.

**Definition 2.4.** *Let  $J$  be the set of equivalence classes of irreducible equivariant Hilbert  $A$ -modules. We define  $\text{Fus}_A(\Gamma)$  to be the based  $\text{Fus}(\Gamma)$ -module with structure coefficients*

$$N_{[V], [\mathcal{E}]}^{[\mathcal{F}]} = \dim(\text{Mor}_A^\Gamma(\mathcal{F}, V \otimes \mathcal{E})).$$

**Lemma 2.5.** *The fusion module  $\text{Fus}_A(\Gamma)$  is cofinite.*

*Proof.* If  $\mathcal{E}$  and  $\mathcal{F}$  are equivariant Hilbert  $A$ -modules, then the space of  $A$ -intertwiners  $\mathcal{L}_A(\mathcal{E}, \mathcal{F}) \subseteq B(\mathcal{E}, \mathcal{F})$  is a unitary corepresentation. This gives rise to the inner product

$$\langle [\mathcal{E}], [\mathcal{F}] \rangle = \sum_{[V] \in I} \dim(\text{Mor}^\Gamma(V, \mathcal{L}_A(\mathcal{E}, \mathcal{F}))[V],$$

which is easily seen to be compatible with the  $\text{Fus}(\Gamma)$ -module structure since

$$\text{Mor}^\Gamma(V, \mathcal{L}_A(\mathcal{E}, \mathcal{F})) \cong \text{Mor}_A^\Gamma(V \otimes \mathcal{E}, \mathcal{F}).$$

□

**Definition 2.6** ([17]). *A discrete quantum group  $\Gamma$  is called torsion-free if any (finite-dimensional) coaction is equivariantly Morita equivalent to a direct sum of trivial coactions.*

More directly, this says a discrete quantum group is torsion-free if and only if any coaction on a finite-dimensional  $C^*$ -algebra is isomorphic to a direct sum of adjoint coactions.

**Definition 2.7.** A discrete quantum group  $\Gamma$  is called *strongly torsion-free* if  $\text{Fus}(\Gamma)$  is torsion-free.

**Theorem 2.8.** A strongly torsion-free discrete quantum group is torsion-free.

*Proof.* Let  $\Gamma$  be a discrete quantum group. Recall that a coaction  $(A, \alpha)$  is called *ergodic* if  $\text{Mor}^\Gamma(\mathbb{1}, A) = \mathbb{C}$ . By decomposing with respect to the fixed point  $C^*$ -algebra  $A^\alpha = \{x \in A \mid \delta(x) = x \otimes 1\}$ , one finds that a general coaction  $\alpha$  is equivariantly Morita equivalent with  $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n \alpha_i)$ , with the  $(A_i, \alpha_i)$  ergodic coactions. Hence to verify torsion-freeness it is sufficient to check that any *ergodic* coaction is isomorphic to an adjoint coaction.

So, let  $A$  be a finite-dimensional  $C^*$ -algebra with an ergodic coaction  $\alpha$  by  $(\mathbb{C}[\Gamma], \Delta)$ . We claim that  $\text{Fus}_A(\Gamma)$  is connected. Indeed, in this case  $A$  itself is an irreducible equivariant Hilbert  $A$ -module, and any equivariant Hilbert  $A$ -module  $\mathcal{E}$  is contained in  $\mathcal{E} \otimes A$ , considered as the tensor product of the unitary corepresentation  $\mathcal{E}$  and the equivariant Hilbert  $A$ -module  $A$ , by the adjoint of the module map  $\mathcal{E} \otimes A \rightarrow A$ .

Thus  $\text{Fus}_A(\Gamma)$  is a non-zero connected cofinite fusion module for  $\text{Fus}(\Gamma)$ . Hence, if we assume  $\Gamma$  is strongly torsion-free, it is isomorphic to  $\text{Fus}(\Gamma)$  as a based  $\text{Fus}(\Gamma)$ -module.

It follows that there exists an irreducible  $A$ -equivariant Hilbert module  $\mathcal{E}$  with  $\langle [\mathcal{E}], [\mathcal{E}] \rangle = \mathbb{1}$ , that is,  $\mathcal{L}_A(\mathcal{E}) = \mathbb{C}$ . But, by ergodicity of  $\alpha$  and invariance and faithfulness of  $\varphi_A$ , we have for  $\xi \neq 0$  that

$$(\text{id} \otimes \varphi)(\langle \alpha(\xi), \alpha(\xi) \rangle_{A \otimes \mathbb{C}[\Gamma]}) = \varphi_A(\langle \xi, \xi \rangle_A) 1 \neq 0,$$

hence  $\langle \mathcal{E}, \mathcal{E} \rangle_A = A$ . It follows that  $\mathcal{E}$  establishes an equivariant Morita equivalence between  $A$  and the trivial coaction on  $\mathbb{C}$ , hence  $(A, \alpha)$  is isomorphic to an adjoint coaction.  $\square$

**Corollary 2.9.** The following discrete quantum groups are strongly torsion-free, and hence torsion-free.

1. The dual of a free unitary quantum group.
2. The free product of strongly torsion-free discrete quantum groups.

*Proof.* It is well-known that the fusion ring of a free product is the free product of the fusion rings associated to the factors [27]. Hence, by Theorem

1.25, a free product of strongly torsion-free discrete quantum groups is again torsion-free.

For the first point, notice that, by [28], a general free unitary quantum group (in the sense of [24]) is a free product of free unitary quantum groups of the form  $A_u(F)$  [3], which are either function algebras on the circle group and hence have fusion ring  $\mathbb{Z}_\mathbb{Z}$ , or else have fusion ring  $A(2)$  [3]. The duals of free unitary quantum groups are thus free by combining the second part of the corollary with Proposition 1.19 and Theorem 1.24.  $\square$

**Remark 2.10.** *For the free orthogonal quantum groups of [24], one has to be more careful. Indeed, the free orthogonal quantum group associated to a one-dimensional matrix is the torsion group  $\mathbb{Z}_2$ , and can by [28] appear as a (free) component in a general free orthogonal quantum group. Nevertheless, if the fundamental representation of the free orthogonal quantum group is irreducible of dimension bigger than 2, its associated fusion ring is  $A(1)$  [2]. So, in this case, the dual discrete quantum group is strongly torsion-free by Proposition 1.23. We hence find back in a combinatorial way a result which was proven algebraically in [26].*

Corollary 2.9 leaves open the question as to whether the free product of torsion-free discrete quantum groups is again torsion-free. This question has a positive answer, but its proof is more natural within the context of module  $C^*$ -categories, which will be treated in the next section, see Theorem 3.16.

To end this section, we prove that strong torsion-freeness is also preserved by Cartesian products. For this, we will need the following lemma.

**Lemma 2.11.** *Let  $\Gamma$  be a discrete quantum group, and assume  $\Gamma$  has a non-trivial finite discrete quantum subgroup. Then  $\Gamma$  is not torsion-free.*

*Proof.* Let  $\mathbb{C}[\Lambda] \subseteq \mathbb{C}[\Gamma]$  be a finite-dimensional Hopf  $C^*$ -subalgebra. Then

$$\Delta : \mathbb{C}[\Lambda] \rightarrow \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda] \subseteq \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Gamma]$$

determines an ergodic coaction of  $\Gamma$ . Assume that it is isomorphic to an adjoint coaction associated to a corepresentation  $(V, \delta)$ . Then the counit on  $\mathbb{C}[\Lambda]$  gives a character on  $B(V)$ , hence  $V$  is one-dimensional and  $\mathbb{C}[\Lambda] = \mathbb{C}$ .  $\square$

**Corollary 2.12.** *Let  $\Gamma_1$  and  $\Gamma_2$  be strongly torsion-free discrete quantum groups. Then  $\Gamma_1 \times \Gamma_2$  is also strongly torsion-free.*

*Proof.* If  $\Gamma$  is a discrete quantum group, a finite fusion subring of  $\text{Fus}(\Gamma)$  necessarily arises from a finite quantum subgroup of  $\Gamma$ . As the fusion ring of a Cartesian product of discrete quantum groups is the tensor product of the fusion rings associated to the components [27], the corollary follows from Theorem 1.26 and Lemma 2.11.  $\square$

### 3 Torsion-freeness for tensor $C^*$ -categories

In this section, we introduce torsion-freeness in the general setting of rigid tensor  $C^*$ -categories [15, 19]. We always assume that our tensor categories are strict with irreducible unit.

**Definition 3.1.** *Let  $(\mathcal{C}, \otimes)$  be a rigid tensor  $C^*$ -category. A  $Q$ -system [15] in  $\mathcal{C}$  (or  $C^*$ -algebra internal to  $\mathcal{C}$ ) consists of an associative algebra  $(A, m, \eta)$  in  $\mathcal{C}$  with  $m : A \otimes A \rightarrow A$  the multiplication and  $\eta : \mathbb{1} \rightarrow A$  the (non-trivial) unit, and such that moreover  $m$  is co-isometric.*

*We call  $A$  ergodic if  $\dim(\text{Mor}(\mathbb{1}, A)) = 1$ .*

**Example 3.2.** *Let  $X$  be an object of  $(\mathcal{C}, \otimes)$ . Let  $R : \mathbb{1} \rightarrow \overline{X} \otimes X$  and  $\overline{R} : \mathbb{1} \rightarrow X \otimes \overline{X}$  be solutions to the conjugate equations. Then  $X \otimes \overline{X}$  is a  $Q$ -system by*

$$m = \frac{1}{\|R\|}(\text{id} \otimes R^* \otimes \text{id}), \quad \eta = \|R\|\overline{R}.$$

*It is ergodic if and only if  $X$  is irreducible.*

If  $\Gamma$  is a discrete quantum group, any coaction  $(A, \alpha)$  on a finite-dimensional  $C^*$ -algebra can be made into a  $Q$ -system by Lemma 2.2. Conversely, any  $Q$ -system arises in this way. We first introduce some terminology which will be needed also later on.

**Definition 3.3.** *Let  $A = (A, m, \eta)$  be a  $Q$ -system in a rigid tensor  $C^*$ -category  $(\mathcal{C}, \otimes)$ . A unitary (right) module for  $A$  is a right  $A$ -module  $(X, n)$  in  $(\mathcal{C}, \otimes)$  with  $n : X \otimes A \rightarrow X$  a co-isometry.*

One similarly introduces unitary left modules and unitary bimodules.

The following proposition is probably well-known, but we could not find a direct proof in the literature.

**Proposition 3.4.** *Let  $\Gamma$  be a discrete quantum group. Then each  $Q$ -system in  $\text{Corep}(\Gamma)$  arises from a coaction of  $(\mathbb{C}[\Gamma], \Delta)$ .*

*Proof.* If  $(A, m, \eta)$  is a  $Q$ -system in  $\text{Corep}(\Gamma)$ , this immediately gives us a unital algebra  $A$  with a coaction  $\alpha : A \rightarrow A \otimes \mathbb{C}[\Gamma]$ . It remains to show that  $A$  is a  $C^*$ -algebra and that  $\alpha$  is  $*$ -preserving.

Since  $A$  is also a finite dimensional Hilbert space, we can make sense of the adjoint  $L_a^*$  of the operator of left multiplication with  $a \in A$ . Define then

$$a^* = L_a^* 1.$$

Since  $m$  is an  $A$ -bimodule map, and since the adjoint of an  $A$ -bimodule map is still an  $A$ -bimodule map by identity (d) in [15, Section 6], we find

$$\begin{aligned} \langle a, b^* c \rangle &= \langle m^* a, L_b^* 1 \otimes c \rangle = \langle (b \otimes 1) m^* a, 1 \otimes c \rangle \\ &= \langle m^*(ba), c \rangle = \langle ba, c \rangle = \langle a, L_b^* c \rangle. \end{aligned}$$

Hence  $L_a^* = L_{a^*}$ , and  $A$  is a  $C^*$ -algebra.

Write  $\varphi_A(a) = \langle 1, a \rangle$ . Then we see  $\varphi_A(a^* b) = \langle a, b \rangle$ , and hence  $\varphi_A$  is a faithful positive functional on  $A$ . Moreover, since  $\alpha$  is a unitary corepresentation, we find

$$(\varphi_A \otimes \text{id})(\alpha(a)^* \alpha(b)) = \varphi_A(a^* b) 1.$$

Hence

$$U : A \otimes \mathbb{C}[\Gamma] \rightarrow A \otimes \mathbb{C}[\Gamma], \quad a \otimes g \mapsto \alpha(a)(1 \otimes g)$$

is unitary. An easy computation shows  $U(L_a \otimes 1)U^* = L_{\alpha(a)}$ , and hence  $\alpha(a^*) = \alpha(a)^*$ .  $\square$

From now on, by module we will mean *right* module.

**Lemma 3.5.** *Let  $A = (A, m, \eta)$  be an ergodic  $Q$ -system. Then the category of unitary  $A$ -modules is a  $C^*$ -category.*

The lemma remains true for arbitrary  $Q$ -systems, but we will not need this result.

*Proof.* The only thing which is not clear is if the adjoints of  $A$ -linear morphisms are again  $A$ -linear. The argument for this is similar to [15, Lemma 6.1].

Namely, fix a unitary module  $(X, n)$  for  $A = (A, m, \eta)$ . Recall that, by means of the standard solutions, one can form *partial traces* [15, Section 3], [19, Section 2.5]

$$(\text{id} \otimes \text{Tr}_Y) : \text{End}(X \otimes Y) \rightarrow \text{End}(X)$$

which are faithful, completely positive  $\text{End}(X)$ -linear maps. Now as  $m^*m$  is an  $A$ -bimodule map from  $A \otimes A$  to  $A \otimes A$ , it follows that  $T = (\text{id} \otimes \text{Tr}_A)m^*m$  is a (left)  $A$ -module map  $A \rightarrow A$ . Hence  $T = m(\text{id}_A \otimes (T \circ \eta))$ . By ergodicity however,  $T \circ \eta = c\eta$  for some  $c > 0$ , and so  $T = c\text{id}_A$ . On the other hand,  $\text{Tr}_{A \otimes A}(m^*m) = \text{Tr}_A(mm^*) = \text{Tr}_A(\text{id}_A)$ , so  $c = 1$ .

Now a small computation reveals that, with  $r = (1_X \otimes m)(n^* \otimes 1_A) - n^*n$ ,

$$r^*r = (n \otimes 1_A)(1_X \otimes m^*m)(n^* \otimes 1_A) - n^*n \geq 0.$$

As

$$\begin{aligned} \text{Tr}_{X \otimes A}((n \otimes 1_A)(1_X \otimes m^*m)(n^* \otimes 1_A) - n^*n) \\ = \text{Tr}_X(nn^*) - \text{Tr}_{X \otimes A}(n^*n) = 0, \end{aligned}$$

we deduce that  $r = 0$ , and so

$$(1_X \otimes m)(n^* \otimes 1_A) = n^*n.$$

We deduce that an  $f \in \text{End}(V)$  lies in  $\text{End}_A(V)$  if and only if  $n(f \otimes 1)n^* = f$ , and hence  $\text{End}_A(V)$  is a  $*$ -algebra.  $\square$

**Corollary 3.6.** *The category of unitary right modules is a (strict, left)  $(\mathcal{C}, \otimes)$ -module  $C^*$ -category [7] by*

$$V \otimes (X, n) = (V \otimes X, \text{id}_V \otimes n).$$

A  $Q$ -system of the form  $X \otimes \overline{X}$  will be called *trivial*.

**Definition 3.7.** *A rigid tensor  $C^*$ -category will be called torsion-free if each ergodic  $Q$ -system is trivial.*

This definition is compatible with the definition for discrete quantum groups by Proposition 3.4. It is also not hard to show that in a torsion-free rigid  $C^*$ -category, every  $Q$ -system is isomorphic to a direct sum of trivial  $Q$ -systems.

Recall that to any rigid tensor  $C^*$ -category  $(\mathcal{C}, \otimes)$  can be associated the fusion ring  $\text{Fus}(\mathcal{C})$ , determined as in Definition 2.1, with the dimension function now given by the intrinsic dimension [15].

**Definition 3.8.** *We call a rigid tensor  $C^*$ -category  $(\mathcal{C}, \otimes)$  strongly torsion-free if  $\text{Fus}(\mathcal{C})$  is torsion-free.*

We will show that Theorem 2.8 still holds in this setting. Note that associated to a (strict) module  $C^*$ -category  $(\mathcal{D}, \otimes)$  one has the based  $\text{Fus}(\mathcal{C})$ -module  $\text{Fus}(\mathcal{D}) = (\mathbb{Z}_J, \otimes)$  where  $J$  is the set of equivalence classes of irreducible objects in  $\mathcal{D}$ .

**Definition 3.9.** *A module  $C^*$ -category  $(\mathcal{D}, \otimes)$  over  $(\mathcal{C}, \otimes)$  is called cofinite (resp. connected) if  $\text{Fus}(\mathcal{D})$  is a cofinite (resp. connected) based  $\text{Fus}(\mathcal{C})$ -module.*

**Lemma 3.10.** *Assume  $(\mathcal{D}, \otimes)$  is a module  $C^*$ -category over  $(\mathcal{C}, \otimes)$ , and assume  $\text{Fus}(\mathcal{D}) \cong \text{Fus}(\mathcal{C})$  as based modules. Then  $(\mathcal{D}, \otimes) \cong (\mathcal{C}, \otimes)$  as module  $C^*$ -categories.*

*Proof.* By assumption, we can find an element  $\mathbb{1} \in \mathcal{D}$  such that  $X \otimes \mathbb{1}$  is irreducible for each  $X$ . As the functor  $X \mapsto X \otimes \mathbb{1}$  also essentially surjective, it follows that  $X \rightarrow X \otimes \mathbb{1}$  is an equivalence of  $C^*$ -categories. As we assume strictness, it is immediate that this is a (strict) equivalence of module  $C^*$ -categories.  $\square$

**Lemma 3.11.** *A rigid tensor  $C^*$ -category is torsion-free if and only if each non-trivial connected cofinite module  $C^*$ -category is equivalent to the tensor  $C^*$ -category as a module  $C^*$ -category.*

*Proof.* Let  $(\mathcal{C}, \otimes)$  be a rigid tensor  $C^*$ -category, and let  $(\mathcal{D}, \otimes)$  be a non-trivial connected cofinite module  $C^*$ -category. Recall the internal Mor-functor of [22] to make sense of  $\underline{\text{Mor}}(M, N) \in \mathcal{C}$  for objects  $M, N \in \mathcal{D}$ , defined by the identity

$$\text{Mor}_{\mathcal{C}}(X, \underline{\text{Mor}}(M, N)) \cong \text{Mor}_{\mathcal{D}}(X \otimes M, N).$$

Then with  $M_0 \in \mathcal{D}$  a fixed irreducible object,  $A = \underline{\text{End}}(M_0)$  is in a natural way an ergodic  $Q$ -system [13, Lemma 2.18]. Moreover, by [22, Theorem 1],  $(\mathcal{D}, \otimes)$  is equivalent to the module  $C^*$ -category  $(\mathcal{D}', \otimes)$  of right unitary  $A$ -modules by means of the correspondence

$$M \mapsto \underline{\text{Mor}}(M_0, M), \quad \underline{\text{Mor}}(M_0, M) \otimes \underline{\text{End}}(M_0) \rightarrow \underline{\text{Mor}}(M_0, M).$$



Here the assumption of finiteness in [22, Theorem 1] is easily seen to be replaceable by cofiniteness. Also, the unitarity of the above module is easily proven by a direct argument involving 2-C\*-category language, as is the fact that the above is then a \*-functor and hence a \*-equivalence.

If now  $(\mathcal{C}, \otimes)$  is torsion-free, it follows that  $A \cong X \otimes \overline{X}$ , and we then have  $(\mathcal{C}, \otimes) \cong (\mathcal{D}', \otimes)$  as module C\*-categories by  $Y \mapsto Y \otimes \overline{X}$ .

Conversely, let  $A$  be an ergodic  $Q$ -system in  $(\mathcal{C}, \otimes)$ , and let  $(\mathcal{D}, \otimes)$  be the associated module C\*-category of unitary  $A$ -modules. It is easy to see that we have an isomorphism of  $Q$ -systems  $A \cong \underline{\text{End}}_A(A)$ , corresponding to the map  $m \in \text{Mor}_A(A \otimes A, A)$ . By assumption,  $(\mathcal{D}, \otimes)$  is equivalent to  $(\mathcal{C}, \otimes)$  as a module C\*-category. Let  $X \in \mathcal{C}$  be the irreducible object corresponding to the irreducible unitary  $A$ -module  $A$  under this equivalence. Then

$$A \cong \underline{\text{End}}_A(A) \cong \underline{\text{End}}_1(X) \cong X \otimes \overline{X}.$$

□

**Theorem 3.12.** *A rigid tensor C\*-category is torsion-free if it is strongly torsion-free.*

*Proof.* By Lemma 3.11, it is sufficient to prove that each non-trivial connected cofinite module C\*-category is isomorphic to  $(\mathcal{C}, \otimes)$ . However, this follows immediately from Lemma 3.10. □

**Remark 3.13.** *It is known that  $\mathbb{Z}[\phi]$  appears as the fusion ring of a (unique) rigid tensor C\*-category, the Fibonacci tensor C\*-category [18, 21]. By Proposition 1.22 and Theorem 3.12, it follows that this (finite!) tensor C\*-category is strongly torsion-free. In fact, this tensor C\*-category is the representation category of the even part of the quantum group at 5th root of unity  $SU(2)_3$ , and one can show the even parts  $SU(2)_{N,\text{even}}$  of the quantum groups at root of unity  $SU(2)_N$  are strongly torsion-free at all odd levels  $N$ , using for example the fusion ring homomorphism  $\text{Fus}(SU(2)_N) \rightarrow \text{Fus}(SU(2)_{N,\text{even}})$  which sends the spin  $\frac{k}{2}$  representation for odd  $k$  to the spin  $\frac{N-k}{2}$ -representation and which is the identity on the integer spin representations, together with the classification of based  $\text{Fus}(SU(2)_N)$ -modules in [10].*

Unlike the case of fusion rings, there also exist non-trivial finite torsion-free tensor C\*-categories with integer dimension function.

**Example 3.14.** Consider the semion category, that is, the rigid tensor  $C^*$ -category  $(\mathcal{C}, \otimes)$  of super-Hilbert spaces  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  with the ordinary tensor product but with the non-trivial associator

$$a_{\mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_k} = (-1)^{ijk}.$$

Then with  $C_2$  the finite simple group of order two,  $\text{Fus}(\mathcal{C}) = \text{Fus}(C_2) = \mathbb{Z}_I$  with  $I = \{+, -\}$  where  $+$  corresponds to  $\mathbb{1}$  and  $- \otimes - = +$ , and with trivial involution.

If  $\mathcal{M}$  is a non-trivial connected cofinite module  $C^*$ -category over  $\mathcal{C}$ , and  $J = \text{Fus}(\mathcal{M})$ , then  $C_2$  acts transitively on  $J$ , and so  $|J| = 1$  or  $|J| = 2$ .

If  $|J| = 1$ , then  $\mathcal{M}$  is the category  $\text{Hilb}$  of Hilbert spaces, which means we have a concrete representation of  $(\mathcal{C}, \otimes)$  inside  $\text{Hilb}$ . This is a contradiction [18, Appendix E], [9, Example 2.3].

Hence  $|J| = 2$  and  $C_2$  acts nontrivially on  $J$ . This means  $(\mathbb{Z}_J, \otimes) \cong (\mathbb{Z}_I, \otimes)$  as based modules, and hence  $(\mathcal{C}, \otimes) \cong (\mathcal{M}, \otimes)$  as module  $C^*$ -categories.

**Remark 3.15.** In fact, this tensor  $C^*$ -category is the representation category of the quantum group at 3rd root of unity  $SU(2)_1$ , and by arguments as in the classification of ‘quantum subgroups of  $SU(2)_N$ ’ [20, 14, 8] one can see that all quantum groups at root of unity  $SU(2)_N$  are torsion-free at all odd levels  $N$ . Nevertheless, they are not strongly torsion-free by the existence of the non-trivial fusion ring homomorphism  $\text{Fus}(SU(2)_N) \rightarrow \text{Fus}(SU(2)_{N,\text{even}})$  (see Remark 3.13).

We do not have any examples however of torsion-free discrete quantum groups which are not strongly torsion-free.

As an application of the theory developed in this section, we can give an easy proof of the following theorems.

**Theorem 3.16.** Let  $\{\Gamma_s \mid s \in S\}$  be a collection of torsion-free discrete quantum groups. Then  $\Gamma = *_{s \in S} \Gamma_s$  is torsion-free.

*Proof.* Let  $(\mathcal{C}, \otimes) = \text{Corep}(\Gamma)$ , and let  $(\mathcal{D}, \otimes)$  be a non-trivial connected, cofinite module  $C^*$ -category. Write  $(\mathbb{Z}_I, \otimes) = \text{Fus}(\Gamma)$ , and  $(\mathbb{Z}_J, \otimes) = \text{Fus}(\mathcal{D})$ . As mentioned already,  $\text{Fus}(\Gamma) = *_{s \in S} \text{Fus}(\Gamma_s)$ .

Let  $X$  be an irreducible object in  $\mathcal{D}$ , and let  $(\mathcal{D}_s^X, \otimes)$  be the (connected, cofinite) module  $C^*$ -category for  $(\mathcal{C}_s, \otimes) = \text{Corep}(\Gamma_s)$  generated by  $X$ . Then,

in the notation of the proof of Theorem 1.25,  $\text{Fus}(\mathcal{D}_s^X) = (B_s^{[X]}, \otimes)$ . However, by assumption of the torsion-freeness of  $\mathbf{\Gamma}_s$ , we have  $(\mathcal{D}_s^X, \otimes) \cong (\mathcal{C}_s, \otimes)$  as a module  $C^*$ -category. Hence  $(B_s^{[X]}, \otimes) \cong (\mathbb{Z}_{I_s}, \otimes)$  as a fusion module.

The proof of Theorem 1.25 can now be followed ad verbatim to conclude that  $\text{Fus}(\mathcal{D}) \cong \text{Fus}(\mathcal{C})$  as fusion modules. By Lemma 3.10 and Lemma 3.11, it follows that  $(\mathcal{C}, \otimes)$  is torsion-free, and hence  $\mathbf{\Gamma}$  is torsion-free.  $\square$

**Theorem 3.17.** *Let  $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2$  be torsion-free discrete quantum groups. Then  $\mathbf{\Gamma}_1 \times \mathbf{\Gamma}_2$  is torsion-free.*

*Proof.* Assume  $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2$  are torsion-free. Then one again sees that the proof of Theorem 1.26 can be copied ad verbatim to conclude that either  $\mathbf{\Gamma}_1 \times \mathbf{\Gamma}_2$  is torsion-free or  $\mathbf{\Gamma}_1$  has a non-trivial finite quantum subgroup. However, the latter is excluded by Lemma 2.11.  $\square$

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